MATHEMATICAL INDUCTION

Suppose that the following claim is made: For all natural numbers, *n*, the natural number $n^3 + 2n$ is divisible by 3. Is the claim true or false? To find out you might begin by examining several cases.

n	$n^3 + 2n$	
1	$1^3 + 2 \cdot 1 = 1 + 2 = 3$	True
2	$2^3 + 2 \cdot 2 = 8 + 4 = 12$	True
3	$3^3 + 2 \cdot 3 = 27 + 6 = 33$	True

So far so good. Now check the statement for $n = 4$, 5, and 6. What are your results? Do you want to keep checking?

One thing is clear: You cannot possibly substantiate the claim for all natural numbers *n*. Only a finite number of instances could be checked. What we need is a method of proof which verifies the claim for all natural numbers in one massive gesture. This method is called <u>mathematical induction</u>.

The Theorem of Mathematical Induction states:

For every $n \in N$ let $P(n)$ be a statement that is either true or false.

If:

1. $P(1)$ is true and
2. Whenever $P(k)$ is true, then $P(k+1)$ is also true,

Then $P(n)$ is true for all $n \in N$

Briefly, what the theorem says is that if a proposition can be shown to be true for the first case and can also be shown to be true for the "successor" case of <u>*any*</u> true case, then a generalization of all cases is produced. (NB. If a statement is true for 1, and if it can be shown that the "next" case in a general sense is true, then the statement is true for 2 which makes it true for 3 which makes it true for 4, etc.)

To apply the Theorem of Mathematical Induction in a proof, then, we must do two things: we must verify that $P(1)$ is true; and we must also verify that $P(k+1)$ is true whenever $P(k)$ is true. (We assume the truth of $P(k)$ and show algebraically that $P(k+1)$ is a logical consequence (true statement) of $P(k)$.

We structure our proofs in the following organized form: THE FIVE A'S (for an A in mathematical induction)

ANCHOR: verify the truth of the hypothesis for $n = 1$
ASSUMPTION: state that the hypothesis is true for values up to $n = k$ for some $k \in N$
ASSERTION: state the hypothesis using $n = k + 1$. This is NOT a statement of the truth of some principle YET but rather a statement of WHAT NEEDS TO BE SHOWN IN ORDER TO DEMONSTRATE the "successor" principle.
ALGEBRA: demonstrate computationally that the validity of the assertion using only the assumption and computations using known properties of the real numbers does occur.
ATTAINMENT: (sometimes called the AHA) state as a summary the original hypothesis in the form $\therefore$ "hypothesis" for all $n \in N$.

SOME EXAMPLES OF INDUCTIVE PROOF

Hypothesis
$$1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$$

Anchor
$$1 = \frac{1(1+1)}{2} \text{ is true.}$$

Assumption
$$1 + 2 + 3 + ... + k = \frac{k(k+1)}{2} \text{ for natural numbers up to } k$$

Assertion
$$1 + 2 + 3 + ... + k + (k+1) = \frac{(k+1)(k+1+1)}{2} \text{ must be demonstrated}$$

Algebra
$$1 + 2 + 3 + ... + k \qquad = \frac{k(k+1)}{2}$$
$$+(k+1) = \qquad +(k+1)$$
$$1 + 2 + 3 + ... + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k^2 + k}{2} + \frac{2k+2}{2}$$
$$= \frac{k^2 + k + 2k + 2}{2} = \frac{k^2 + 3k + 2}{2} = \frac{(k+1)(k+2)}{2}$$

Attainment
$$\therefore 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2} \forall n \in N$$

Hypothesis	$n^3 + 2n$ is always divisible by 3 when n is a natural number.
Anchor	$1^3 + 2 \cdot 1 = 1 + 2 = 3$ is divisible by 3.
Assumption	$k^3 + 2k$ is divisible by 3 for values up to k, a natural number.
Assertion	$(k+1)^3 + 2(k+1)$ is divisible by 3 must be demonstrated
Algebra	Examine $(\text{k+1})^3 + 2(k+1)$
	$= k^3 + 3k^2 + 3k + 1 + 2k + 2 = k^3 + 3k^2 + 5k + 3$
	$= (k^3 + 2k) + 3k^2 + 3k + 3 = (k^3 + 2k) + 3(k^2 + k + 1)$
	Since by our initial assumption $(k^3 + 2k)$ is divisible by 3 and since 3 is a factor of the other addend, so it too must be divisible by 3, we have demonstrated that the assertion is true whenever the assumption is true.
Attainment	$\therefore n^3 + 2n$ is always divisible by 3 when n is a natural number.

Hypothesis	$3^n \ge 1 + 2n$
Anchor	$3^1 \ge 1 + 2 \cdot 1 = 3$ is true
Assumption	$3^k \ge 1 + 2k$ for natural numbers up to k.
Assertion	$3^{k+1} \ge 1 + 2(k+1)$ must be demonstrated.
Algebra	Examine 3^{k+1}.
	$3^{k+1} = 3 \cdot 3^k \ge 3(1 + 2k) = 3 + 6k \ge 3 + 2k = 1 + 2(k+1)$
	Thus, $3^{k+1} \ge 1 + 2(k+1)$
Attainment	$\therefore 3^n \ge 1 + 2n \ \forall n \in N$

Exercises (extra practice if you wish) PROVE (using mathematical induction):

1. $1+3+5+...+(2n-1)=n^2$
2. $2+4+6+...+2n=n(n+1)$
3. $5+7+9+...+\left[5+(2n-1)\right]=\frac{n\left[10+2(n-1)\right]}{2}$
4. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+...+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$
5. $1+3+6+...+\frac{n(n+1)}{2}=\frac{n(n+1)(n+2)}{6}$
6. $1^3+2^3+3^3+...n^3=\frac{n^2(n+1)^2}{4}=\left[\frac{n(n+1)}{2}\right]^2$
7. $1+4+7+...+(3n-2)=\frac{n(3n-1)}{2}$
8. $1+2(2)+3(2)^2+...+n(2)^{n-1}=1+(n-1)2^n$
9. $2^x \geq 1+x$
10. n^3+3n^2+2n is evenly divisibly by 6 for all natural numbers.
11. $2n \leq 2^n$
12. $2^{n-1} \leq n!$