MOTIVATING THE PRODUCT AND QUOTIENT RULES

(with grateful thanks to Paul Foerster)

Having learned the power rule, we have a vehicle for motivating students to develop the product and quotient rules for derivatives for themselves.

Start with a simple power function in which the exponent is a fairly large even number.

Eg., $f(x) = x^{24}$. What is the derivative? $f'(x) = 24x^{23}$.

Let's consider several different factorizations of x^{24} for which both exponents of the factors are prime numbers.

$f(x) = x^5 x^{19} = x^7 x^{17} = x^{13} x^{11}$ to name several. Let's just start with the first one.

$f(x) = x^5 x^{19}$ If there is a "product rule" for finding derivatives, what sorts of values might it involve?

Let $\begin{matrix} u(x) = x^5 & v(x) = x^{19} \\ u'(x) = 5x^4 & v'(x) = 19x^{18} \end{matrix}$. How are we going to obtain the same answer with our "product rule" that we obtain from the power rule? (We know that we have to get the same answer both ways or our product rule won't be valid.)

Students will probably notice that the coefficients of the derivatives of the factors do add up to the coefficient of the derivative of the function, but the exponents are all wrong. Encourage them to explore what is missing in each term to resolve the exponent inconsistency. It will not take them terribly long to realize that if each of the derivatives of the factors is multiplied by the other factor and then the two partial products are added, they get the right answer.

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Having learned the power rule, we have a vehicle for motivating the development of the product and quotient rules for derivatives.

Let's start with a simple power function in which the exponent is a fairly large even number.

Eg., $f(x) = x^{24}$. What is the derivative?

Let's consider several different factorizations of x^{24} for which both exponents of the factors are prime numbers.

$f(x) = x^{—} \cdot x^{—}$

If there is a "product rule" for finding derivatives, what sorts of values might it involve?

Let
$u(x) = x^{—} \qquad v(x) = x^{—}$
$u'(x) = \_\_\_\_\_\_\_\_\_ \qquad v'(x) = \_\_\_\_\_\_\_\_\_$

Do you have any ideas how you could get to the correct derivative from these values? How can we obtain the same answer with our "product rule" that we obtain from the power rule? (We do know that we have to get the same answer both ways or our product rule won't be valid.) Experiment with your conjectures below.

Summarize your product rule symbolically. If $f(x) = (u(x)) \cdot (v(x))$ then
$f'(x) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

Try the following problem and see if your rule works for it as well. $f(x) = 8x^{13} = (4x^5) \cdot (2x^8)$

Now take $f(x) = x^{24}$ and rewrite it as a product of three factors (don't worry about having prime exponents this time, but use three different exponents) $f(x) = x^{—} \cdot x^{—} \cdot x^{—}$.
Experiment with these factors and their derivatives to see if you can conjecture an "extended" product rule for more than two factors.

Summarize your conjecture here.
If $f(x) = a(x) \cdot b(x) \cdot c(x)$, then $f'(x) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

Now let's see what we can do about discovering a rule for functions that are quotients of other functions.

Again let's consider a simple power function for which we can compute the answer directly.

Rewrite $f(x) = x^{24}$ as a quotient of two powers for which the exponents are prime numbers, eg., $f(x) = x^{24} = \frac{x^{29}}{x^5}$.

Recall that our goal is to determine that the derivative of this quotient is $f'(x) = 24x^{23}$

Experiment with some relationships that you can form between the functions that form the numerator and denominator and their derivatives to try to find a relationship that will give the correct answer.

Summarize your conjecture here. If $f(x) = \frac{a(x)}{b(x)}$ then $f'(x) =$ ____________________

See if your rule gives the correct derivative for $f(x) = 8x^{17} = \frac{48x^{20}}{6x^3}$