THE CROSS PRODUCT

$\vec{a}\times\vec{b}=\vec{c}$ where the vector $\vec{c}$ is a vector **perpendicular** to the plane of $\vec{a}$ and $\vec{b}$ and with **magnitude** equal to the **area** of the **parallelogram** that is formed by $\vec{a}$ and $\vec{b}$. (We will look at the proof of the area formula another day).

So $\vec{c}$ is a normal vector to the plane of $\vec{a}$ and $\vec{b}$, and is thus normal to each of $\vec{a}$ and $\vec{b}$.

Let's figure out some of the cross products for the simple basis vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$ and then we will figure out how to use these basic facts to compute cross products for more complicated vectors, and what applications the cross product has for helping us to solve more interesting and complex problems.

The right hand rule for the cross product tells us how to interpret both the vector and its sign. Point the thumb of your right hand in the direction of the first factor, and point the index finger of your right hand in the direction of the second factor. The palm will be facing (positive or negative) in the direction of the cross product and if you "flap" your fingers, the direction of the flapping will show which basis vector the product is (alternatively, if you point your pinky perpendicular to your palm in the ONLY direction it will move, that will be the cross product vector.

$\times$	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$			
$\vec{j}$			
$\vec{k}$			

Because of the fact that the magnitude of the cross product is the area of the parallelogram formed, we see that any vector crossed with itself will give a "degenerate" parallelogram and thus have no area so the cross product of any vector with itself is the null ($\vec{0}$) vector.

Now to an example and the TWO ways to compute its cross product.

$$\left(3\vec{i}-5\vec{j}+\vec{k}\right)\times\left(2\vec{i}+3\vec{j}+6\vec{k}\right)$$

$$\begin{aligned}
&\left(3\vec{i}-5\vec{j}+\vec{k}\right)\times\left(2\vec{i}+3\vec{j}+6\vec{k}\right)\\
&=3\vec{i}\times2\vec{i}+3\vec{i}\times3\vec{j}+3\vec{i}\times6\vec{k}+-5\vec{j}\times2\vec{i}+-5\vec{j}\times3\vec{j}+-5\vec{j}\times6\vec{k}+\vec{k}\times2\vec{i}+\vec{k}\times3\vec{j}+\vec{k}\times6\vec{k}\\
&=6\left(\vec{i}\times\vec{i}\right)+9\left(\vec{i}\times\vec{j}\right)+18\left(\vec{i}\times\vec{k}\right)+-10\left(\vec{j}\times\vec{i}\right)+-15\left(\vec{j}\times\vec{j}\right)+-30\left(\vec{j}\times\vec{k}\right)+2\left(\vec{k}\times\vec{i}\right)+3\left(\vec{k}\times\vec{j}\right)+6\left(\vec{k}\times\vec{k}\right)\\
&=6\left(\vec{0}\right)+9\left(\vec{k}\right)+18\left(-\vec{j}\right)-10\left(-\vec{k}\right)-15\left(\vec{0}\right)-30\left(\vec{i}\right)+2\left(\vec{j}\right)+3\left(-\vec{i}\right)+6\left(\vec{0}\right)\\
&=9\vec{k}-18\vec{j}+10\vec{k}-30\vec{i}+2\vec{j}-3\vec{i}\\
&=-33\vec{i}-16\vec{j}+19\vec{k}
\end{aligned}$$

An alternate (and faster and easier to use) method involves setting up what would appear to be a 3 by 3 determinant and computing that.

$$\begin{vmatrix} i & j & k \\ 3 & -5 & 1 \\ 2 & 3 & 6 \end{vmatrix} \begin{matrix} i & j \\ 3 & -5 \\ 2 & 3 \end{matrix} \quad = \quad \left(-30\vec{i}+2\vec{j}+9\vec{k}\right)-\left(-10\vec{k}+3\vec{i}+18\vec{j}\right) = -33\vec{i}-16\vec{j}+19\vec{k}$$

Consider the question of finding a plane given three points: Suppose that the points are: $P(-3,4,1), Q(0,-1,2), R(-1,7,7)$

Let's write parametric equations for the plane and solve by linear combinations:

$$\begin{array}{llll} x=-3+3s+2t & x=-3+3s+2t & y=4-5s+3t & 33x-99z=-198-528t \\ y=4-5s+3t & \underline{-3z=-3-3s-18t} & \underline{5z=5+5s+30t} & \underline{16y+80z=144+528t} \\ z=1+s+6t & x-3z=-6-16t & y+5z=9+33t & 33x+16y-19z=-54 \end{array}$$

SURPRISE!!!

Thus, because the cross product gives the vector normal to the plane, we can use the cross product to find that normal and then can use the trick we have been using for months to actually WRITE the equation for the plane!

Another wonderful application. Suppose you want to know the distance from a point to a line in three space (remember we have done this problem the long way). Since the magnitude of the cross product is the area of the parallelogram, and the area of a parallelogram is base times height, we can use the cross product's magnitude to find the distance of a point from a line (since that distance will BE the height of the parallelogram). Think about how to do this!

Another cool application with another unexpected shortcut for a problem we worked the long way.

Find an equation for a plane through a given point so that the new plane is perpendicular to TWO given planes. Visualize this first.
What special vectors will need to LIE in this new plane? And how can you use those vectors to determine the new plane? And (here is the unexpected extra!) what do you notice about the relationship between the new plane and the line of intersection of the two given planes?