PROOF THAT THE MAGNITUDE OF THE CROSS PRODUCT IS THE AREA OF THE PARALLELOGRAM

Consider two general vectors $\vec{a} = e\vec{i} + f\vec{j} + g\vec{k}$ and $\vec{b} = p\vec{i} + q\vec{j} + r\vec{k}$

So $\vec{a} \times \vec{b} = (fr - gq)\vec{i} + (gp - er)\vec{j} + (eq - pf)\vec{k}$ and

$$\begin{aligned}
\left\|\vec{a} \times \vec{b}\right\| &= \sqrt{(fr - gq)^2 + (gp - er)^2 + (eq - pf)^2} \\
&= \sqrt{f^2r^2 - 2fgqr + g^2q^2 + g^2p^2 - 2egpr + e^2r^2 + e^2q^2 - 2efpq + p^2f^2} \\
&= \sqrt{e^2p^2 + e^2q^2 + e^2r^2 + f^2p^2 + f^2q^2 + f^2r^2 + g^2p^2 + g^2q^2 + g^2r^2 - \left(e^2p^2 + f^2q^2 + g^2r^2 + 2fgqr + 2egpr + 2efpq\right)} \\
&= \sqrt{\left(e^2p^2 + e^2q^2 + e^2r^2 + f^2p^2 + f^2q^2 + f^2r^2 + g^2p^2 + g^2q^2 + g^2r^2\right)\left(1 - \frac{\left(e^2p^2 + f^2q^2 + g^2r^2 + 2fgqr + 2egpr + 2efpq\right)}{e^2p^2 + e^2q^2 + e^2r^2 + f^2p^2 + f^2q^2 + f^2r^2 + g^2p^2 + g^2q^2 + g^2r^2}\right)} \\
&= \sqrt{\left(e^2 + f^2 + g^2\right)\left(p^2 + q^2 + r^2\right)\left(1 - \frac{\left[\left(e\vec{i} + f\vec{j} + g\vec{k}\right)\bullet\left(p\vec{i} + q\vec{j} + r\vec{k}\right)\right]^2}{\left(e^2 + f^2 + g^2\right)\left(p^2 + q^2 + r^2\right)}\right)} \\
&= \left\|\vec{a}\right\|\left\|\vec{b}\right\|\sqrt{1 - \cos^2\theta} \\
&= \left\|\vec{a}\right\|\left\|\vec{b}\right\|\sin\theta
\end{aligned}$$