SUMMARY OF POLYNOMIAL FUNCTION RULES AND TESTS

TERMS AND CONCEPTS:

Definition: A polynomial function of degree n is a function of the form
$p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x^1 + a_0 x^0$, where a_i for $0 \le i \le n$ is a complex number; $a_n \neq 0$ and $n \ge 0$.

If $p(x_i) = 0$ then x_i is a root of $p(x)$. The zeroes of the polynomial $p(x)$ are the values x_i, $0 < i \le n$ such that $p(x_i) = 0$.

If $p(x)$ is a polynomial, then $p(x)$ is everywhere continuous.

RULES, FORMULAE, AND THEOREMS:

Division Algorithm: For any polynomial $p(x)$ and any number *k*, there exists a unique polynomial $q(x)$ and number *r* such that $p(x) = (x-k)q(x) + r$

Remainder Theorem: If a polynomial is divided by $(x-k)$ where *k* is a constant, the remainder from the division is the value of the polynomial $p(k)$ at $x = k$.

Factor Theorem: $(x-k)$ is a factor of the polynomial $p(x)$ if and only if $p(k) = 0$.

Rational Root Theorem: If a rational number $\frac{m}{q}$ is a root of $p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x^1 + a_0 x^0$ where the a_i's are all integers, then *m* is a factor of a_0 (the constant term) and *q* is a factor of a_n (the leading coefficient).

Fundamental Theorem of Algebra: If $p(x)$ is a polynomial of degree *n*, where $n \ge 1$, there exists at least one complex number *k* such that $p(k) = 0$ and $p(x)$ can be factored completely into linear factors: $p(x) = a_n (x - r_1)(x - r_2)...(x - r_n)$ and $p(x)$ has precisely *n* roots although some roots may occur with multiplicities greater than one.

Conjugate Theorem: If $p(x)$ is a polynomial with real coefficients and if there exists some complex number $a + bi, b \neq 0$ so that $p(a+bi) = 0$ then $p(a-bi) = 0$ also (*ie*, complex roots of polynomials with real coefficients occur in conjugate pairs)

Intermediate Value Theorem: (The Intermediate Value Theorem can also be applied to any function $f(x)$ that is continuous on a closed interval $[a,b]$) If $f(a) \neq f(b)$, then for each value of *d* that is between $f(a)$ and $f(b)$, there exists a number *c* with $a < c < b$ so that $d = f(c)$. Moreover, if $p(x)$ is a polynomial with real coefficients and if for real values of *a* and *b*, $p(a)$ and $p(b)$ have opposite signs, then there is at least one zero of $p(x)$ between *a* and *b*.

Descartes Rule of Signs: Let $p(x)$ be a polynomial with real coefficients written in descending order (by powers). The following information about the real roots of $p(x)$ can be determined:

(1) The number of positive real zeroes of $p(x)$ is either equal to the number of sign changes occurring in the coefficients of $p(x)$ or differs from that value by $2k$ (fewer values would occur if there were complex zeroes—hence the differing by $2k$ due to the occurrence of complex zeroes in conjugate pairs)

(2) Construct the polynomial represented by replacing x throughout by $-x$ (ie construct $p(-x)$The number of negative real zeroes of $p(x)$ is either equal to the number of sign changes occurring in the coefficients of $p(-x)$ or differs from that value by $2k$ (for the same reasons as above).

Relationship of the coefficients and roots in polynomials:

Quadratic: If $p(x) = ax^2 + bx + c$ or $p(x) = x^2 + \frac{b}{a}x + \frac{c}{a}$ and r_1 and r_2 are the roots of $p(x)$, then $r_1 + r_2 = -\frac{b}{a}$ (with leading coefficient equal to 1, the sum of the roots is **opposite** of the *x*-coefficient) and $r_1 r_2 = \frac{c}{a}$ (the product of the roots is equal to the constant term).

Generally: Given an *n*th degree polynomial with leading coefficient 1 and roots $r_1, r_2, \ldots r_n$, the sum of the roots is equal to the **opposite** of the coefficient of the x^{n-1}-term, the sum of the "pairwise products" of the roots ($r_i r_j, i < j$ for all pairings of *I* and *j*) is the coefficient of the x^{n-2}-term, and so on (sum of all triple products of roots, quadruple products, etc. with the "odd" numbers of factors type terms being opposites of the coefficients and the "even" numbers of factors type terms being the actual coefficients).

TECHNIQUES:

Synthetic Division: A short way of writing the steps in the long division of a polynomial by a linear factor of the form $x - k$. Synthetic Division produces a suppressed quotient and remainder and consequently also produces the value of the polynomial $p(k)$. When $p(k) = 0$, *k* is a root of $p(x)$ and $x - k$ is a factor of $p(x)$.

Upper and Lower Bounds on Roots: Given $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x^1 + a_0 x^0$ with $a_n > 0$. If, for $k > 0$ in the synthetic division by $x - k$, all values in the bottom line are non-negative, then there are NO real roots larger than *k*. Likewise, if for $k < 0$ in the synthetic division by $x - k$, the values in the bottom line alternate between positive and negative (with zero representing positive or negative as needed if the remainder of the pattern is correct), then there are NO real roots smaller than *k*.

Successive Approximation of Roots: (This technique is made totally obsolete by the availability of the graphing calculator, but the notion is still worth knowing).

(1) Determine two values a and b