HOW TO FIND A RECTANGULAR EQUATION FOR A PLANE
GIVEN THREE NON-COLLINEAR POINTS

We know that if three points are non-collinear they will determine a unique plane. And, further, we know that the rectangular equation of a plane is of the form $Ax + By + Cz = D$ so any plane containing three given points will have to be satisfied by the substitution of the coordinates of each of those three points into the correct equation for the plane.

Thus, if we are given points $P(x_0, y_0, z_0), Q(x_1, y_1, z_1), R(x_2, y_2, z_2)$, then substituting them in turn should give us THREE true statements (equations). Notice, however, that we obtain three equations but there are FOUR unknown quantities $A, B, C,$ and D. This situation is known as a system of Diophantine Equations. There are an infinite number of solutions to this system (because of the fact that we have fewer equations than we have unknowns). But the solutions (once found) are all equivalent to a single solution (they differ by a constant multiplier from one another).

This suggests a method for determining the rectangular equation of a plane if we are given three of its points. I will demonstrate this method using a particular example of three randomly chosen points. (By the way, this method is the first of several, at least three, methods we will learn. Each future method will, in my opinion, be superior to this method, but for the moment, this method is the ONLY one we have the knowledge to use)

Given: $P(5,-1,2), Q(-1,4,-1), R(2,3,6)$

So:
$$\begin{aligned} 5A - 1B + 2C &= D \qquad (1) \\ -1A + 4B - 1C &= D \qquad (2) \\ 2A + 3B + 6C &= D \qquad (3) \end{aligned}$$

This is a system of equations whose solution (any) will produce a rectangular equation of the desired plane. A strategy that will most easily enable us to solve this system will involve solving for $A, B,$ and C all in terms of the fourth variable, D. To do this we want to use linear combinations but we do NOT want to eliminate D.

Looking at the particular system above, it looks as if we could eliminate either A or C pretty easily. Let's do A.

$$\begin{aligned} 5A - 1B + 2C &= D \\ -5A + 20B - 5C &= 5D \\ \hline 19B - 3C &= 6D \end{aligned}$$

and

$$\begin{aligned} -2A + 8B - 2C &= 2D \\ 2A + 3B + 6C &= D \\ \hline 11B + 4C &= 3D \end{aligned}$$

Let's use these results.

$$19B - 3C = 6D \qquad 76B - 12C = 24D$$
$$11B + 4C = 3D \quad \text{produces} \quad 33B + 12C = 9D$$
$$109B = 33D \qquad \boxed{B = \frac{33}{109}D}$$

Having now obtained a "value" for one of the missing values (the coefficients to use for the plane we want), we can just substitute it back into an earlier equation, solve for another variable's value, and then substitute both of them back in to get the third of the values (all, recall, in terms of *D*). Then we can write an equation (sort of).

$$11B + 4C = 3D \qquad 11\left(\frac{33}{109}D\right) + 4C = 3D \qquad 4C = \frac{327 - 363}{109}D \qquad \boxed{C = -\frac{9}{109}D}$$

$$-1A + 4B - 1C = D \qquad -A + \frac{132}{109}D - \left(-\frac{9}{109}D\right) = D \qquad -A = \frac{109 - 132 - 9}{109}D \qquad \boxed{A = \frac{32}{109}D}$$

Thus our equation is: $\frac{32D}{109}x + \frac{33D}{109}y - \frac{9D}{109}z = D$ That is almost what we need, but now we have to get rid of the *D*. The best way to do this is to simply CHOOSE a value for *D* that suits us and the equation. Probably we would prefer if the equation consisted exclusively of INTEGER valued coefficients and constants, so our choice for *D* will probably want to be a number that will "cancel" out any denominators (they are not always going to be the same value) that are present. In this case the value 109 (in general, the number that represents the least common denominator—least common multiple of the existing denominators) is the smallest number that will accomplish this. Thus, our final result (the equation of the plane containing these three points) is $32x + 33y - 9z = 109$. To verify that this is really the plane we wanted, we could substitute each of the points into this equation and see that all three statements it produces are true!

$$32 \cdot (5) + 33 \cdot (-1) - 9 \cdot (2) = 160 - 33 - 18 = 109$$
$$32 \cdot (-1) + 33 \cdot (4) - 9 \cdot (-1) = -32 + 132 + 9 = 109$$
$$32 \cdot (2) + 33 \cdot (3) - 9 \cdot (6) = 64 + 99 = 54 = 109$$

And one more thing about rectangular planes in general. Because the general equation of such a plane is $Ax + By + Cz = D$, and because any point that is ON a coordinate axis, we can easily find the intercepts of the plane by substituting zero for two of the three variable values and solving what is left. Thus the intercepts for a plane are $x = \frac{D}{A}$ $y = \frac{D}{B}$ $z = \frac{D}{C}$ which also allows us to construct an "intercept" form for the equation: $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ for which the *a*, *b*, and *c* are the numerical values of those intercepts that we solved for above.